U.S. NAVAL RESEARCH LABORATORY

An Analytical and Data-Driven Approach to Self-Consistent Wave-Particle Interaction

Particle Dynamics in the Radiation Belts
AGU-Chapman Meeting

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Wave-form data can improve our understanding of nonlinear wave-particle **interactions**

How?

- Resonant particles control waveforms
- Laboratory experiments
- Bayesian time-series techniques

What?

- Chorus
- Laboratory triggered emissions
- Hiss

Wave electromagnetic field is controlled by particles

Dispersion Relation

$$\frac{c^2k^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{(\omega + \Omega_e)(\omega + \Omega_i)}$$

Polarization

$$E_y = iE_x$$

Non-resonant particles provide

- Restoring force and
- Medium for wave to propagate

Resonant particles provide

- Damping and growth
- Frequency and amplitude modulation

Resonant particles interact with amplitude and phase

$$h = \sum_i^M \left\{ (1-\bar{\omega}) p_{\psi i} + \frac{1}{2}(\bar{k} p_{\psi i} + I_i)^2 - 2\alpha \sqrt{\frac{p_{\psi i} J}{M}} \cos(\psi_i - \theta) + \alpha^2 \frac{J}{M} \right\}$$

$$P = J + \sum_i p_{\psi i}$$

Self-consistent wave-particle Hamiltonian

Conserves Energy

Conserves Momentum

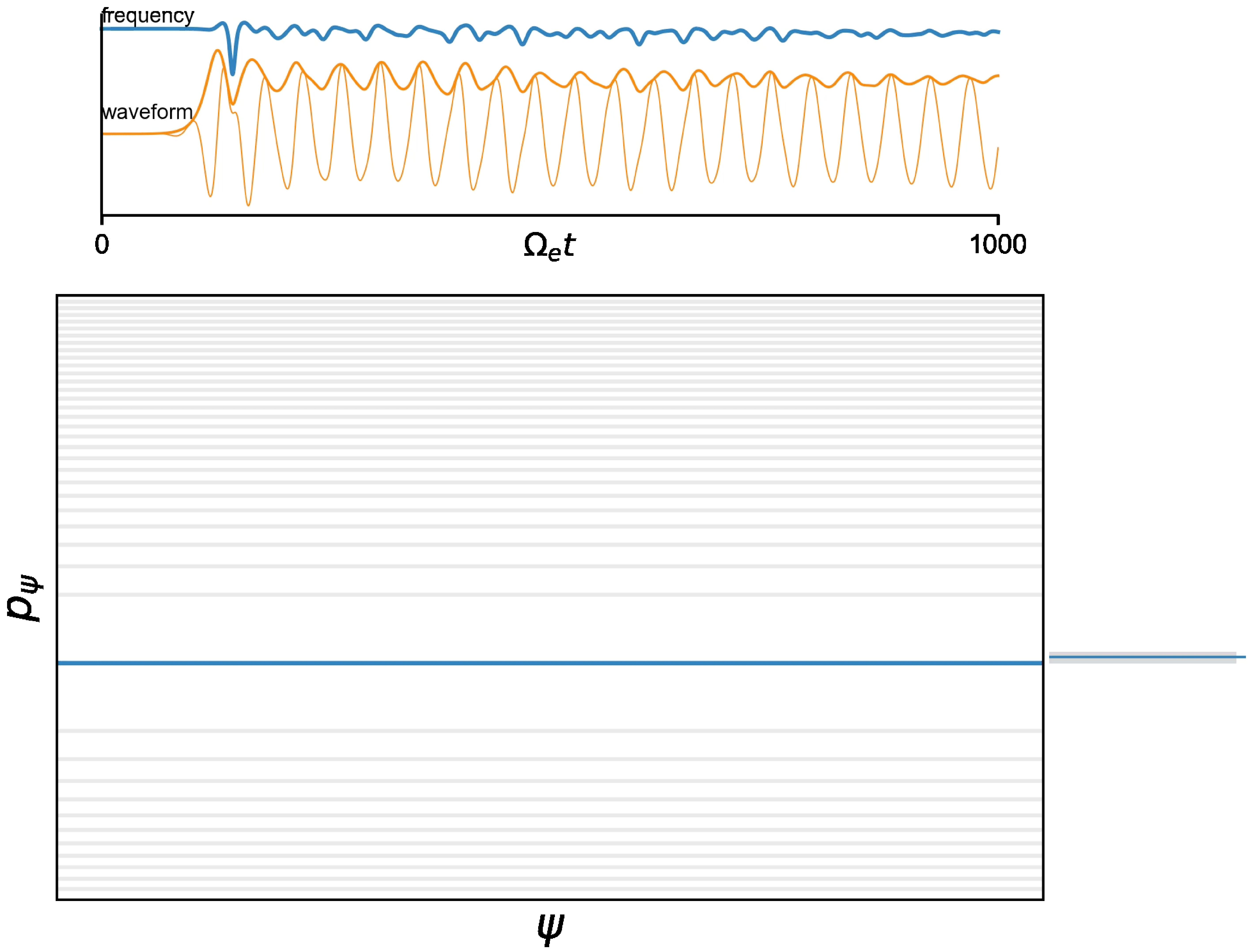
Particle phase ψ_i

Particle momentum $p_{\psi i}$

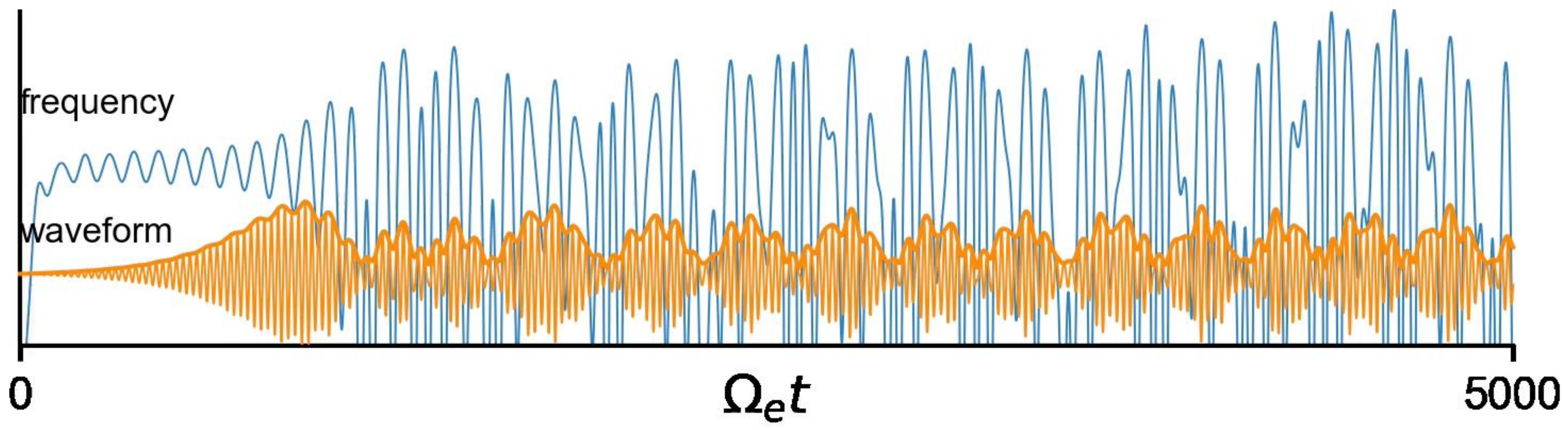
Wave action $J \propto (\text{Amplitude})^2$

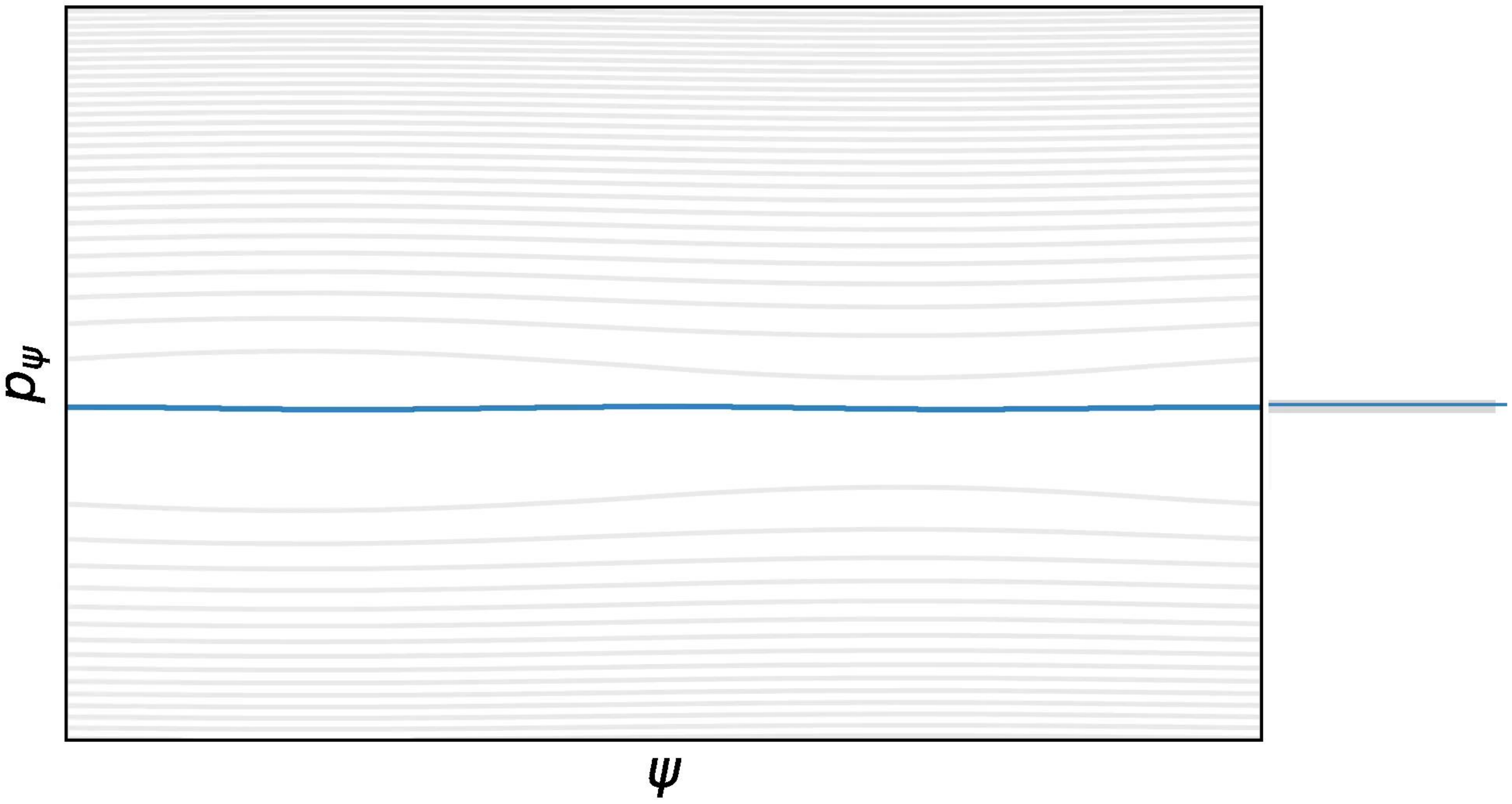
Wave phase $\theta \longrightarrow \dot{\theta} = \delta\omega = \frac{\partial h}{\partial J}$

Resonant particles interact with amplitude and phase

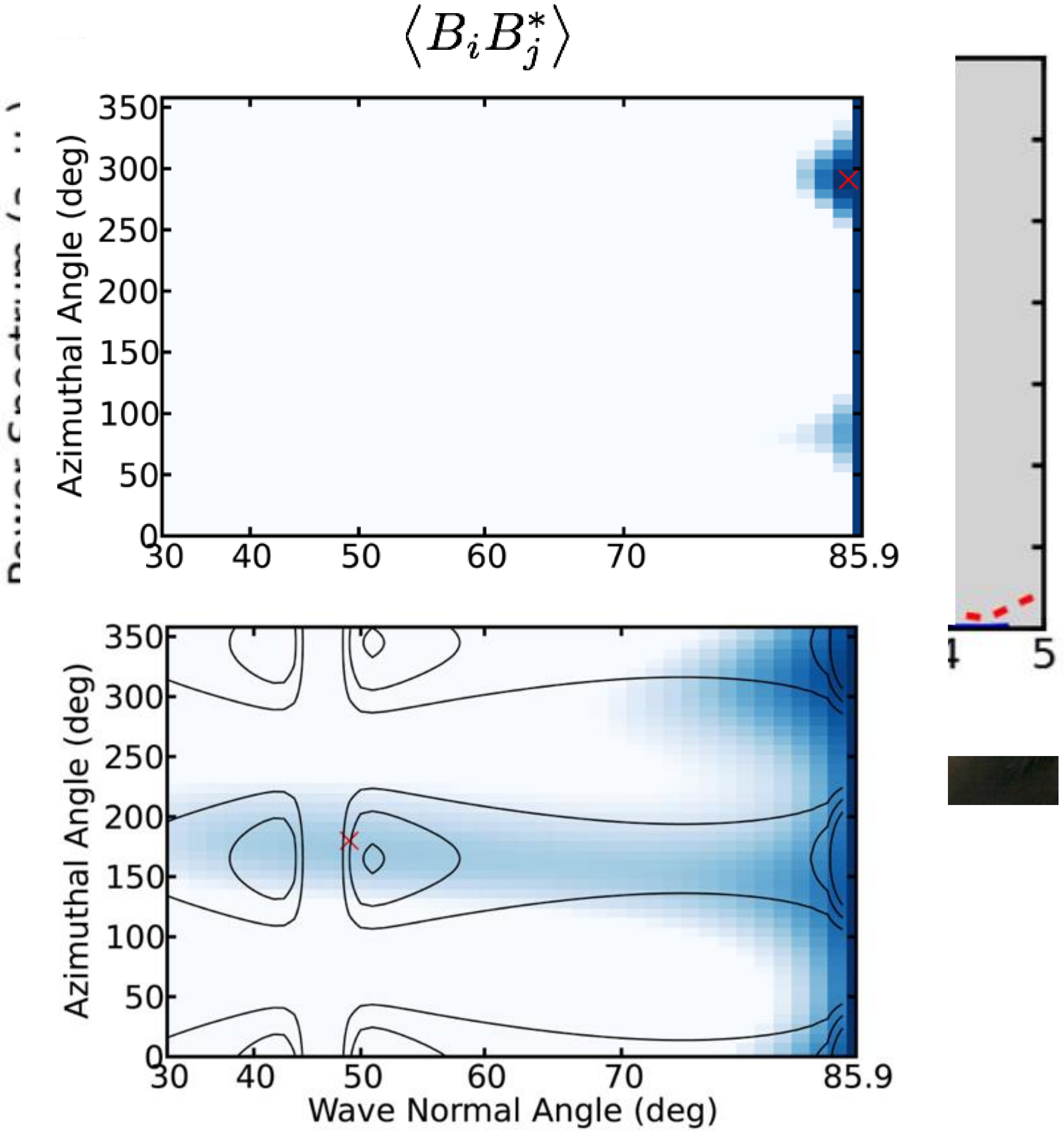


Resonant particles interact with amplitude and phase





Laboratory experiments can test theories and analysis techniques



Control and repeatability allow tests of theory

Nonlinear scattering of VLF waves

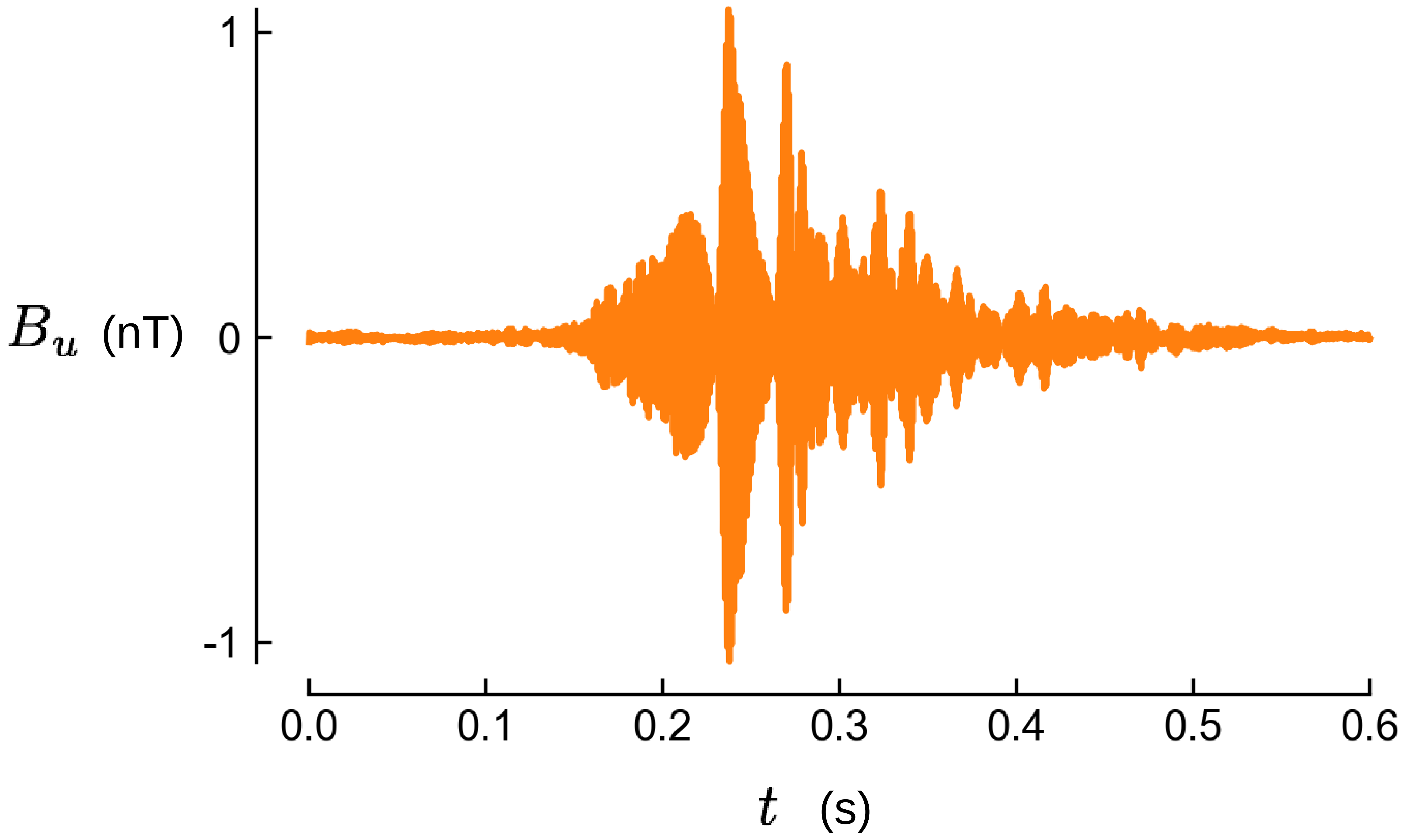
Instabilities

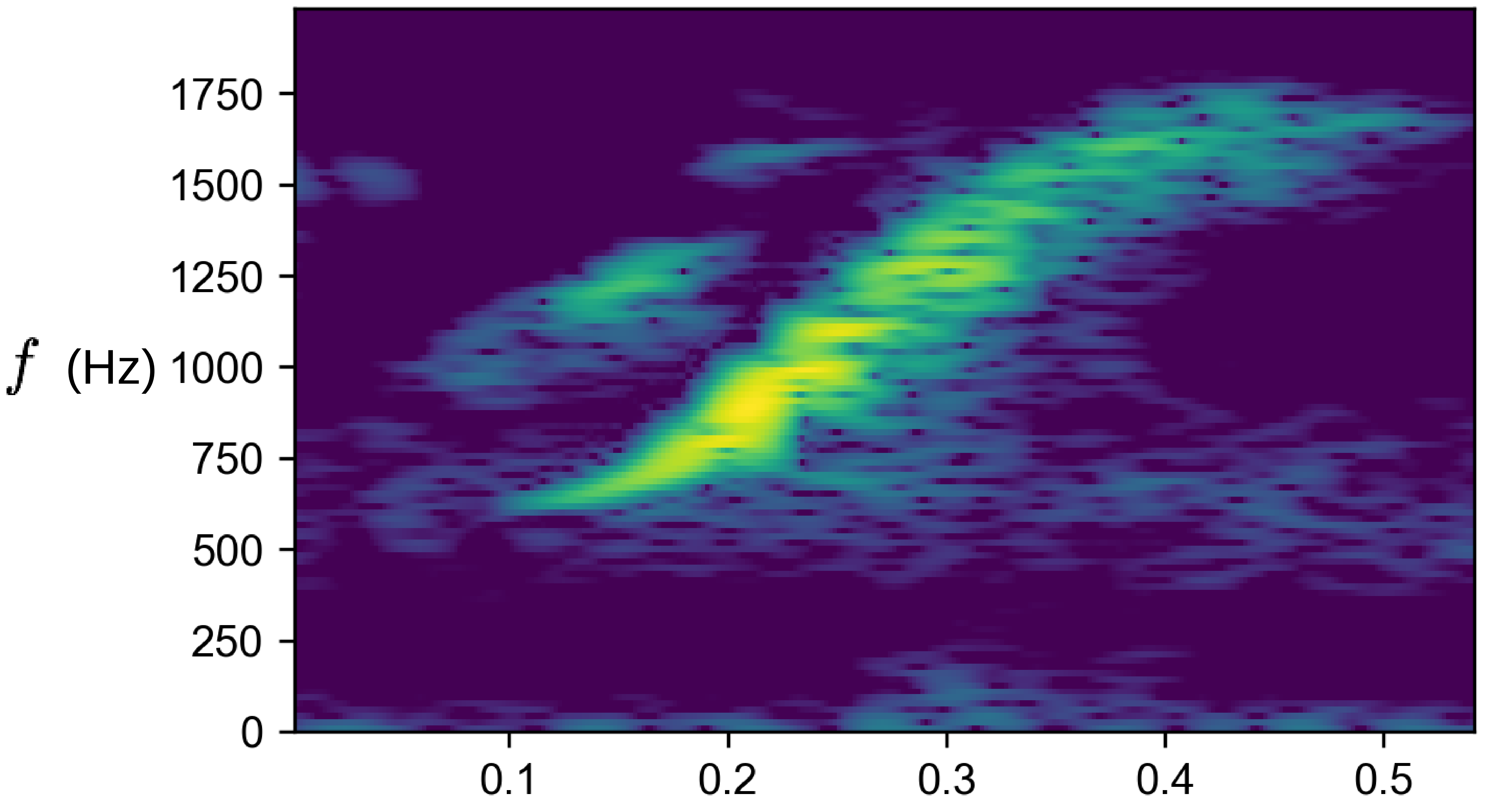
Multiple experimental probes allow

Unambiguous wavelength determination

Validation of single point wavelength estimates

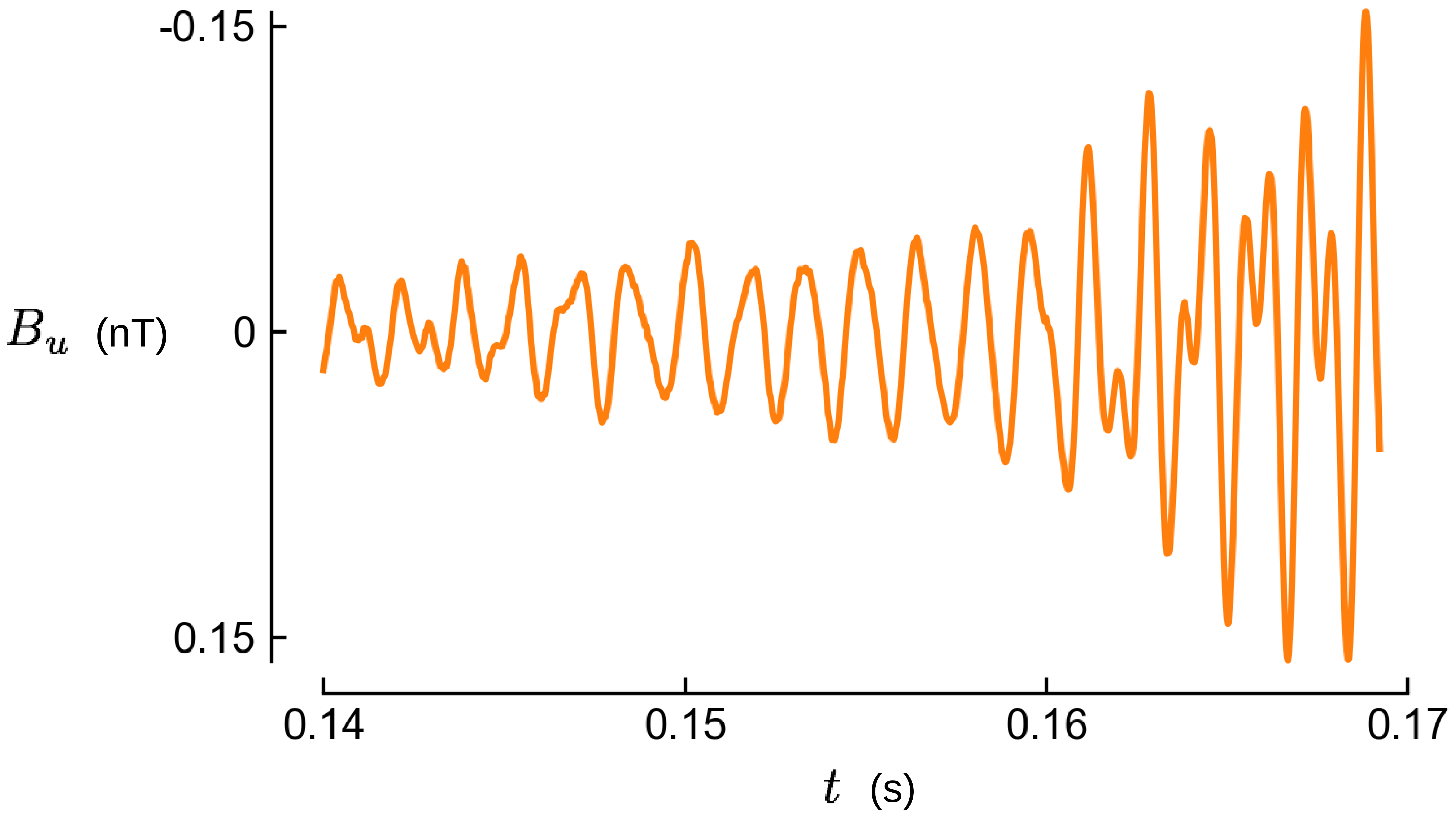
Chorus modulates amplitude and chirps frequency





Bayes' law gives alternative interpretation of Fourier transform

$$f(t) = A\cos(\omega t) + B\sin(\omega t)$$



Bayes' law gives alternative interpretation of Fourier transform

Likelihood

Prior

Posterior

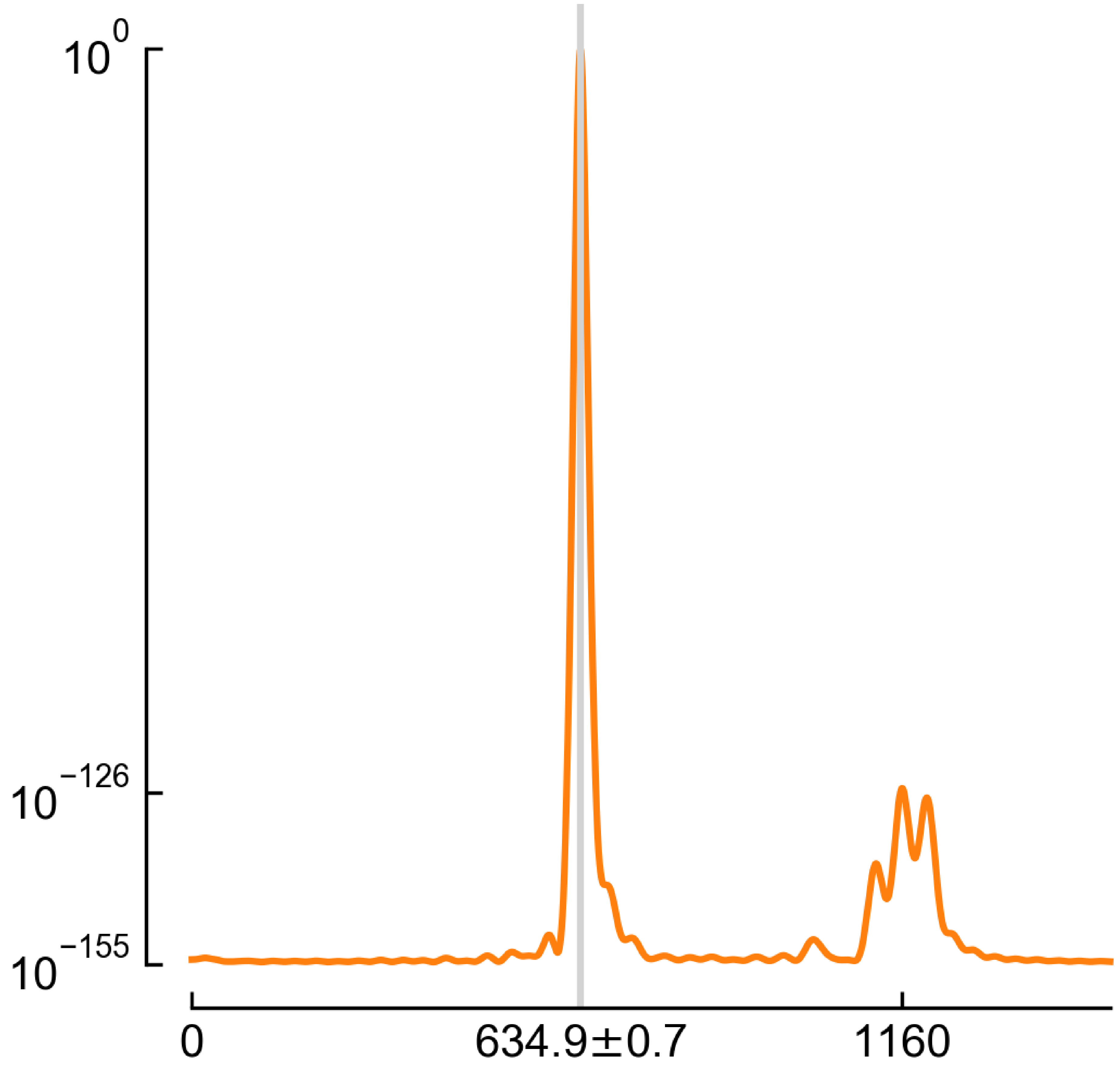
$$P(\omega|D_i, I) = \int P(\omega, A, B|D_i, I)\, dA\, dB = \frac{P(D_i|\omega, A, B, I)\; P(\omega, A, B|I)}{P(D_i|I)}$$

Evidence

$$P(D_i|\omega, A, B, I) \propto \exp\left(-\sum_i \frac{(f(t_i) - D_i)^2}{2\sigma^2}\right)$$

Bayes' law gives alternative interpretation of Fourier transform

$$P(\omega|D_i, I) \propto \exp\left(-\frac{1}{N\sigma^2}\left|\sum_i d_i e^{j\omega t_i}\right|^2\right)$$



Bayesian spectral analysis has several advantages

Amplitude linear

Suppose you have S models

SVD, WDF

wave vector estimation

$$f(t) = \sum_{n} \ldots \sum_{i}^{S} \ldots [illegible] \ldots B e^{\gamma t} \sin(\omega t + \alpha t^2)$$

Frequency nonlinear

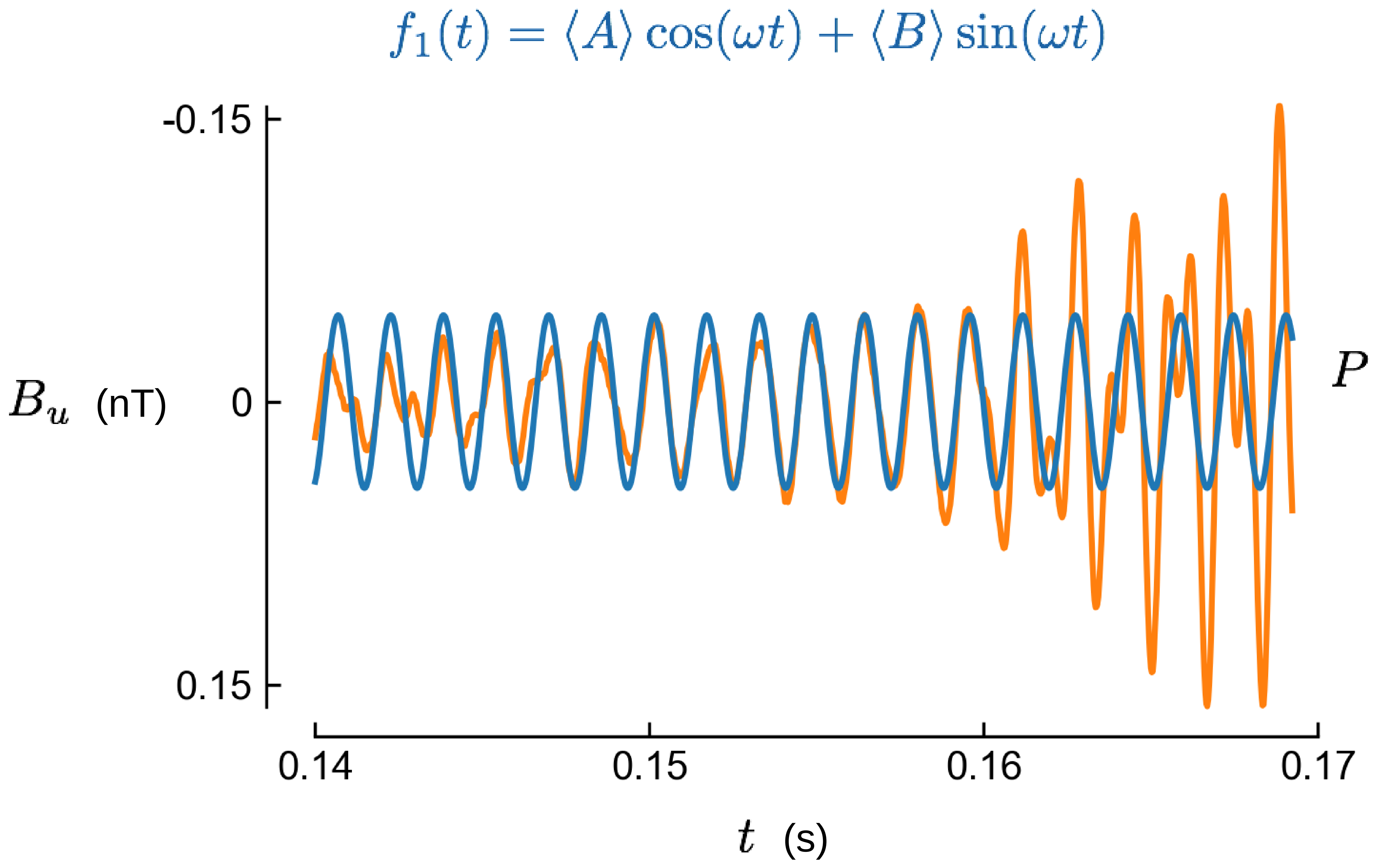
Not limited to single frequency

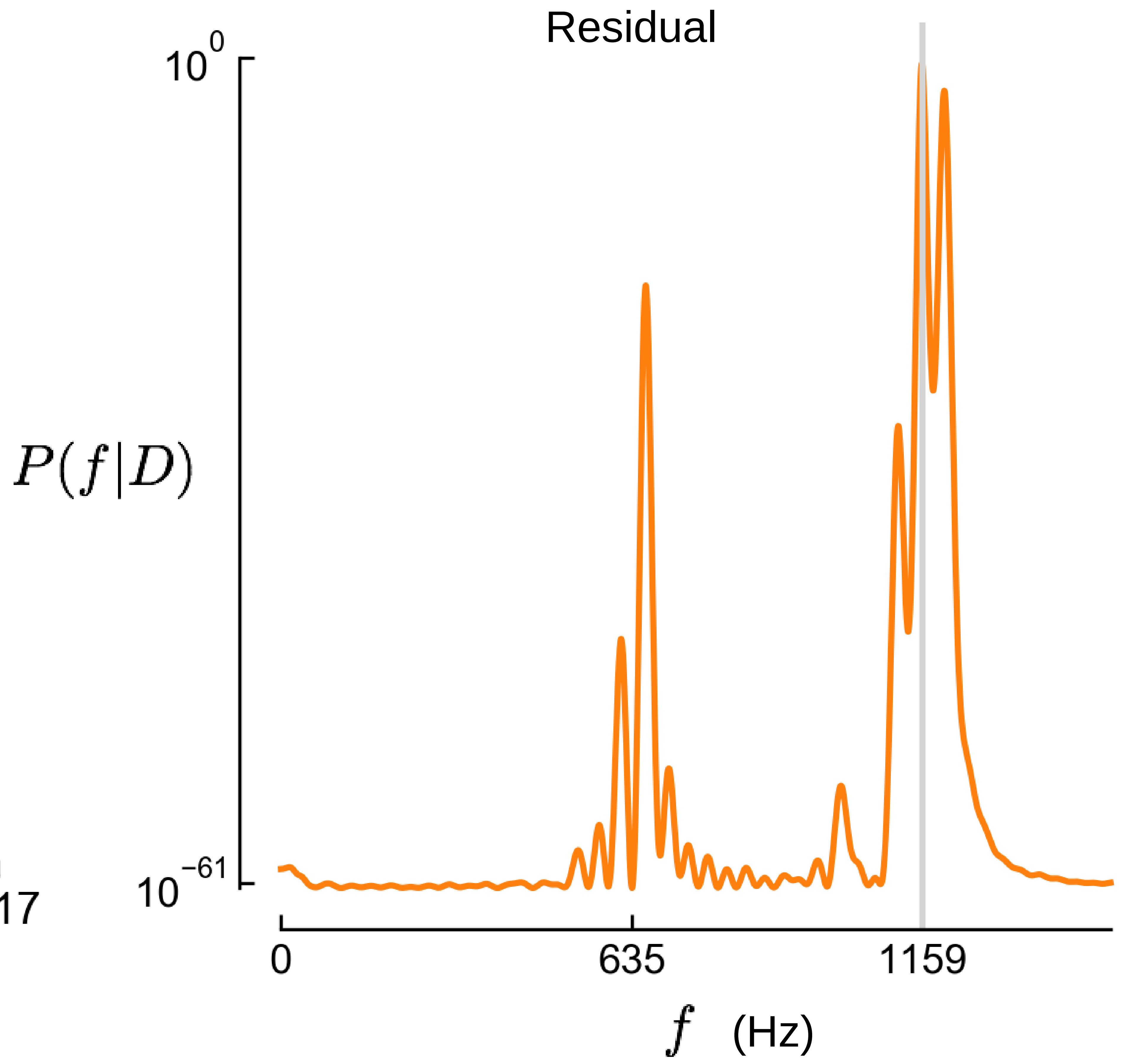
Gives probability distribution directly

- No need for ensemble average
- Allows for model comparison

Can incorporate other data to improve resolution

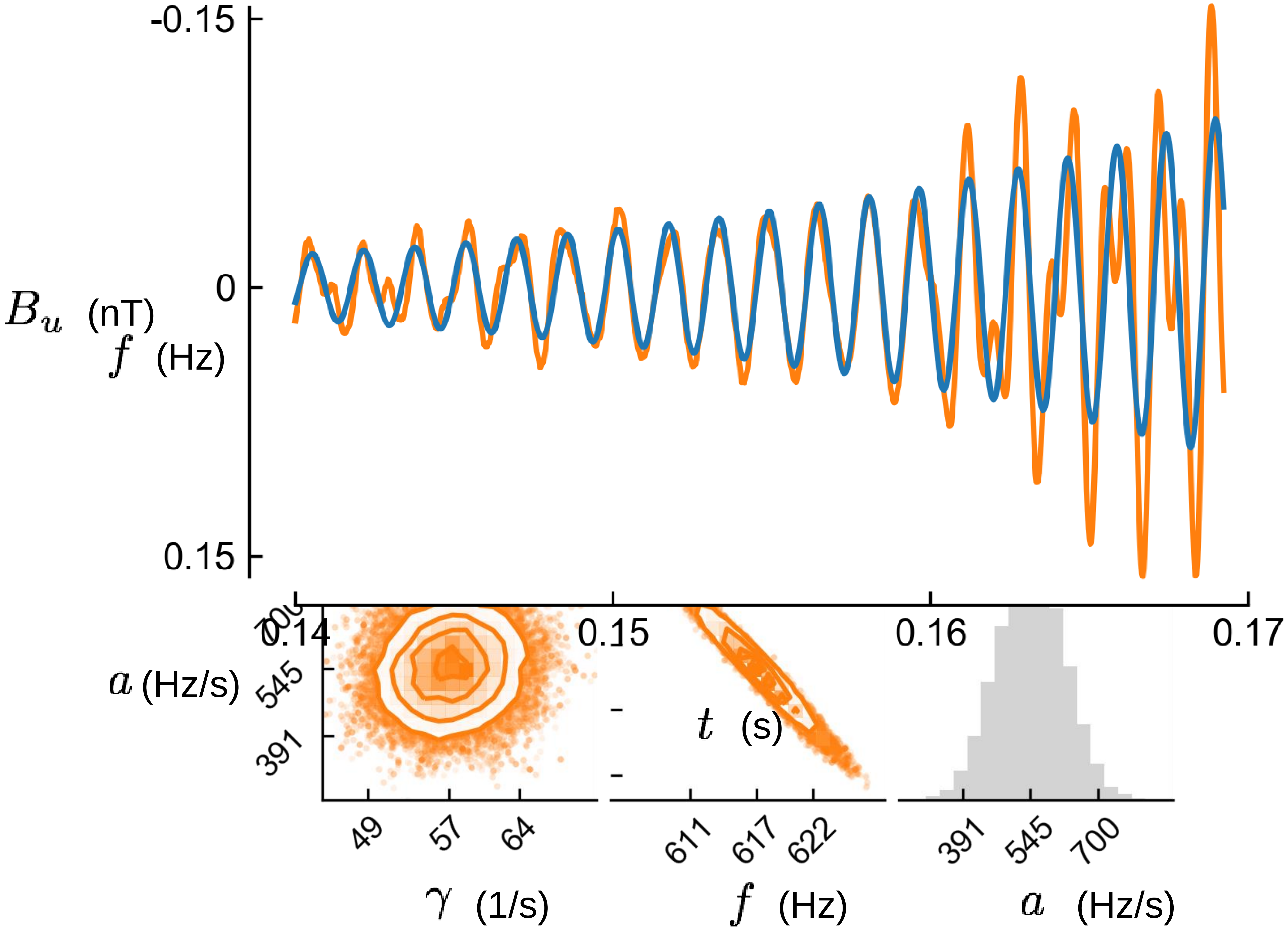
Model with a linear frequency chirp is more probable than a single Fourier mode

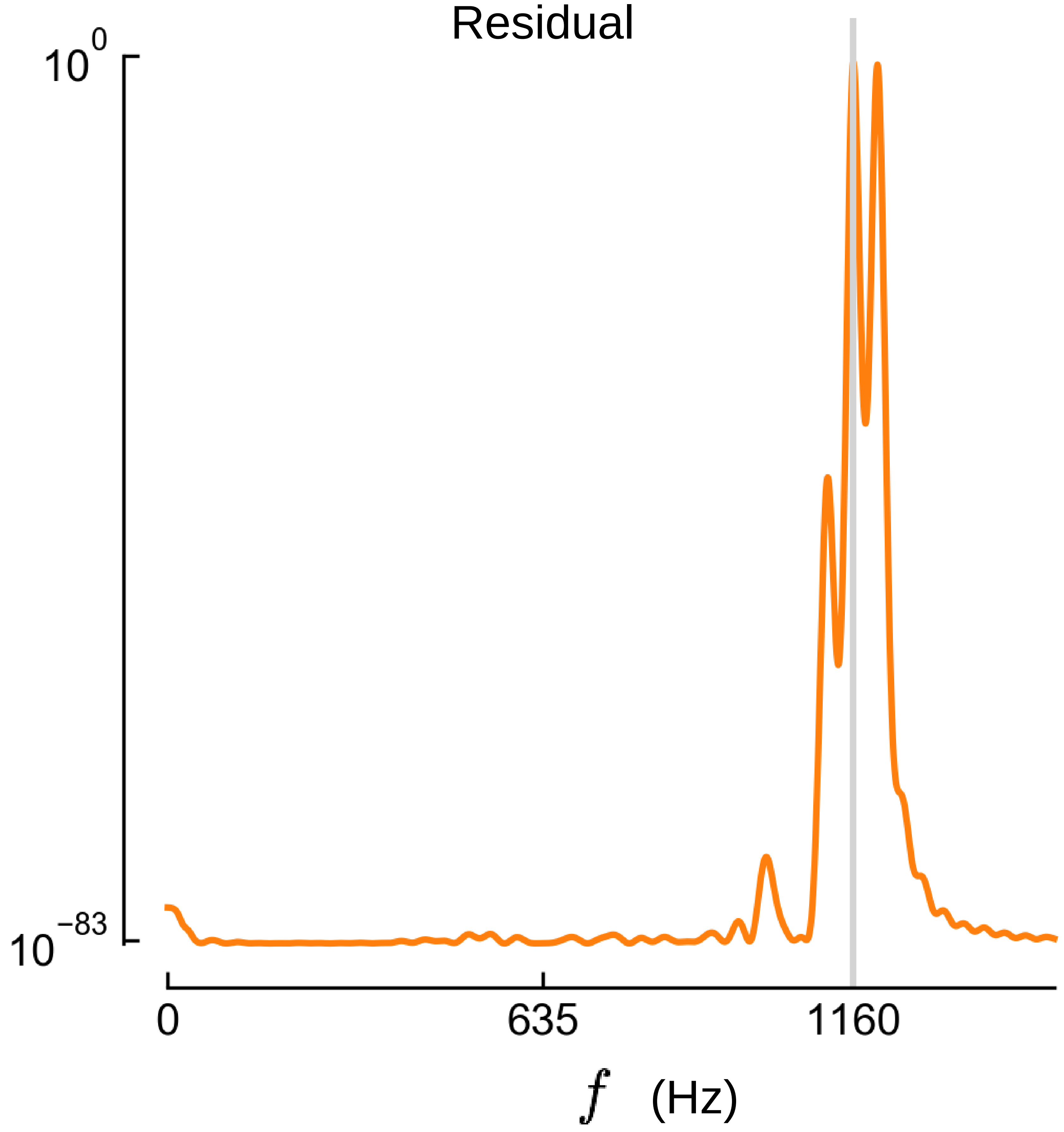




Model with a linear frequency chirp is more probable than a single Fourier mode

$$f_2(t) = A e^{\gamma t}\cos(\omega t + \alpha t^2) + B e^{\gamma t}\sin(\omega t + \alpha t^2)$$

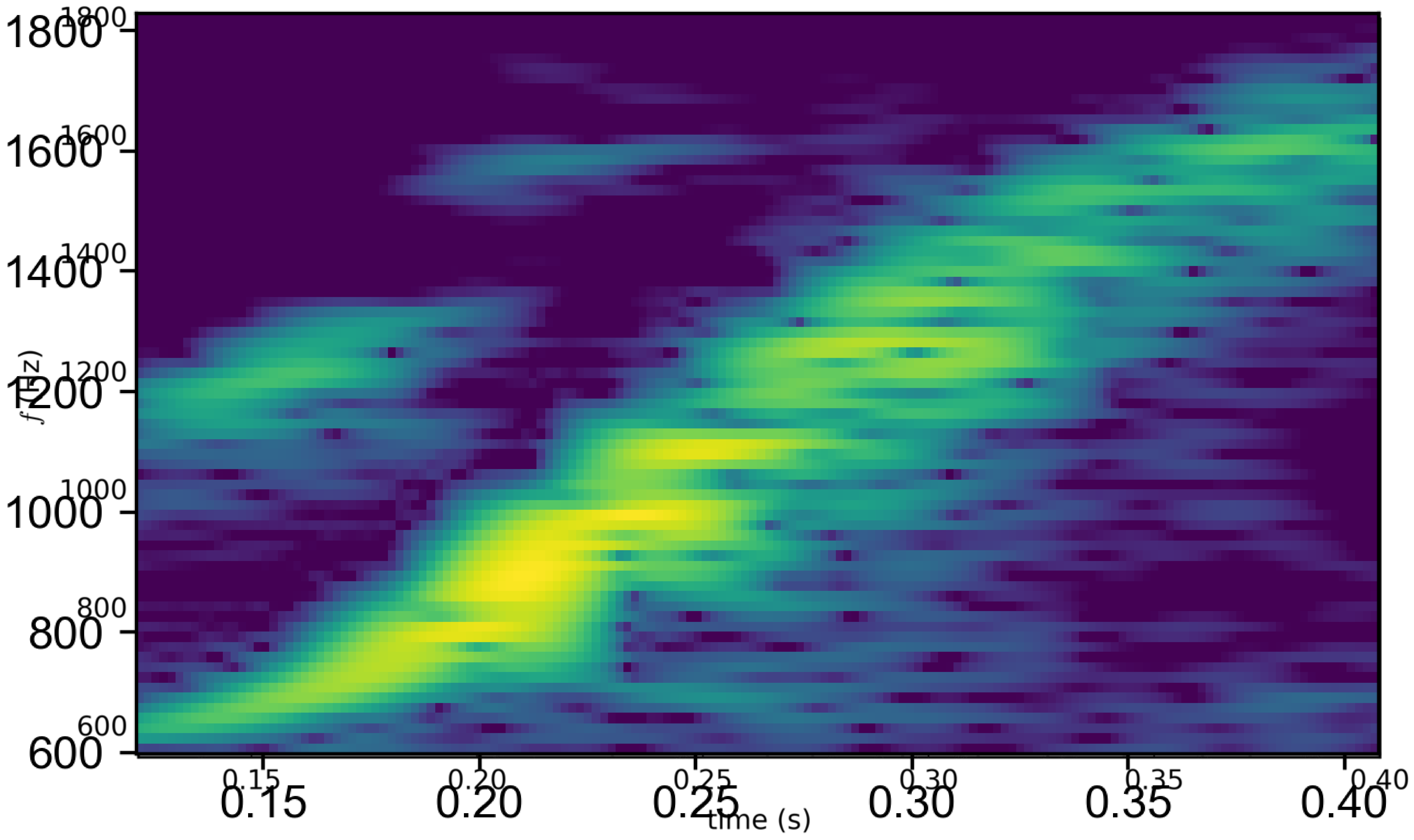




Model with a linear frequency chirp is more probable than a single Fourier mode

Model	Relative Probability
$f_1 = A\cos(\omega t) + B\sin(\omega t)$	$P(f_1 \vert D) = 10^{-64}$
$f_1 = Ae^{\gamma t}\cos(\omega t + \alpha t^2) + Be^{\gamma t}\sin(\omega t + \alpha t^2)$	$P(f_2 \vert D) = 1 - 10^{-64}$

Bayesian analysis of chorus reveals sub-packet boundaries



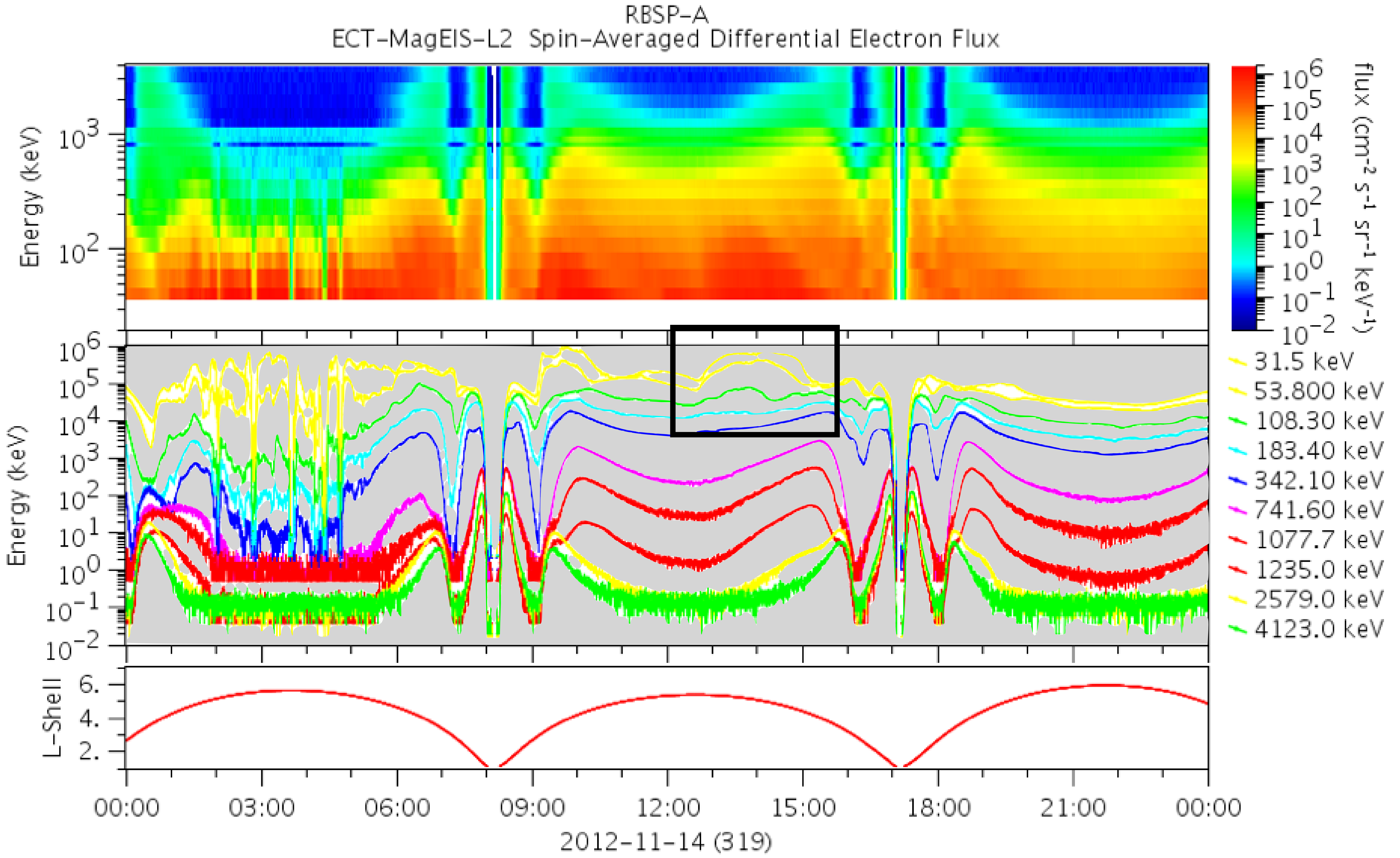
Bayesian analysis of chorus reveals trapping oscillations

$$\omega_t = \sqrt{k v_\perp \Omega_e \frac{B_w}{B}}$$

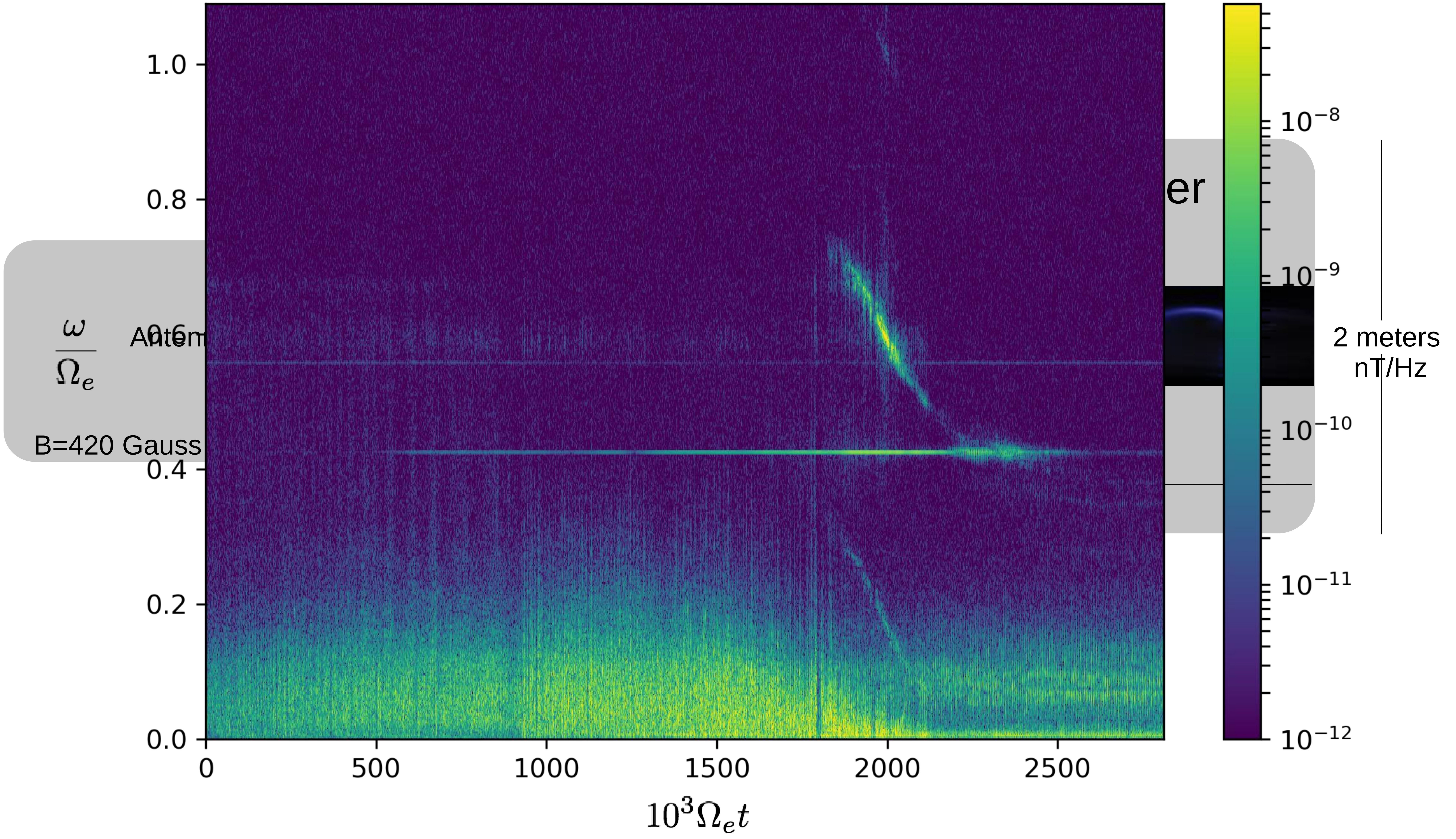
10-15 KeV

$$\omega - k v_\parallel = \Omega_e$$

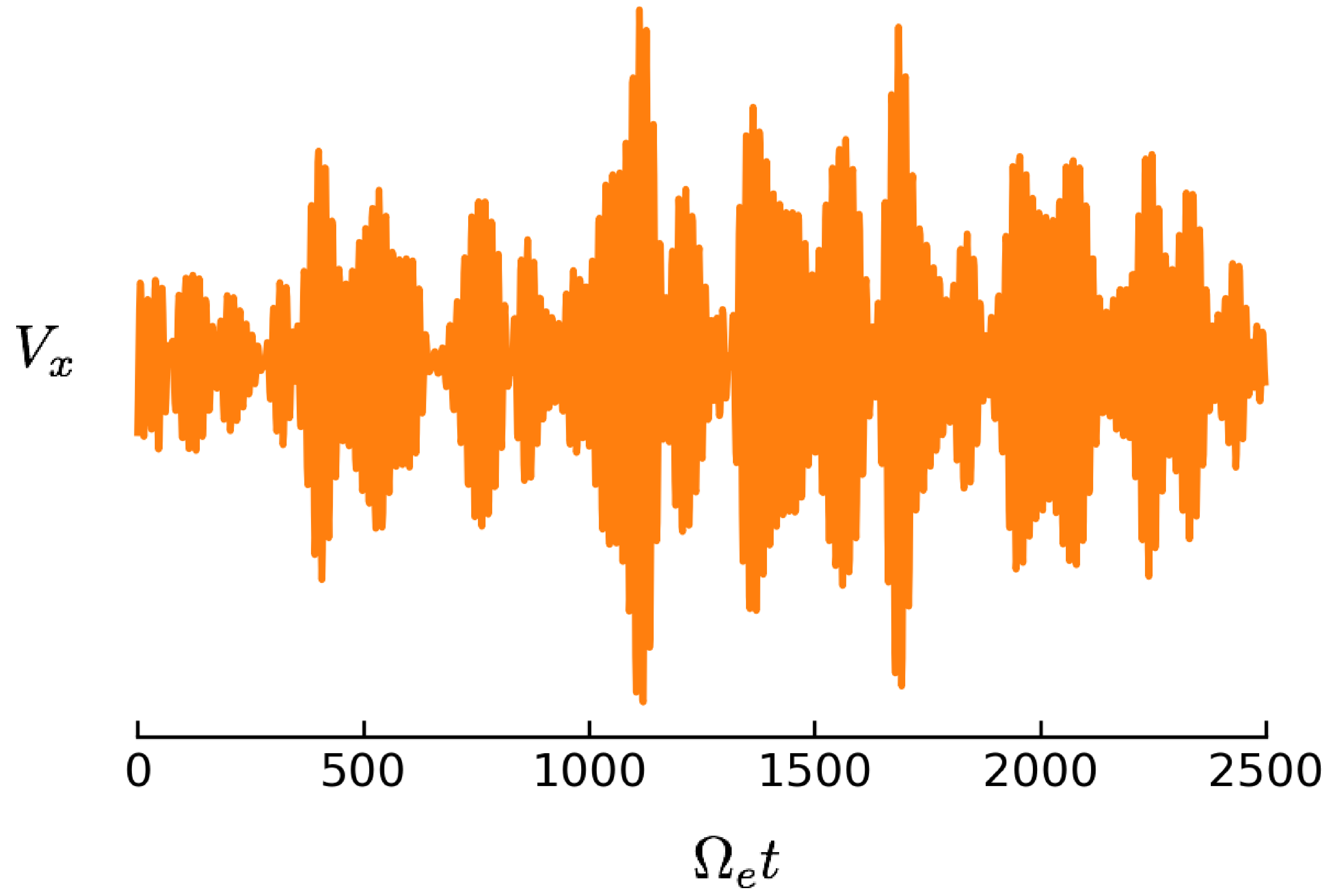
50 KeV

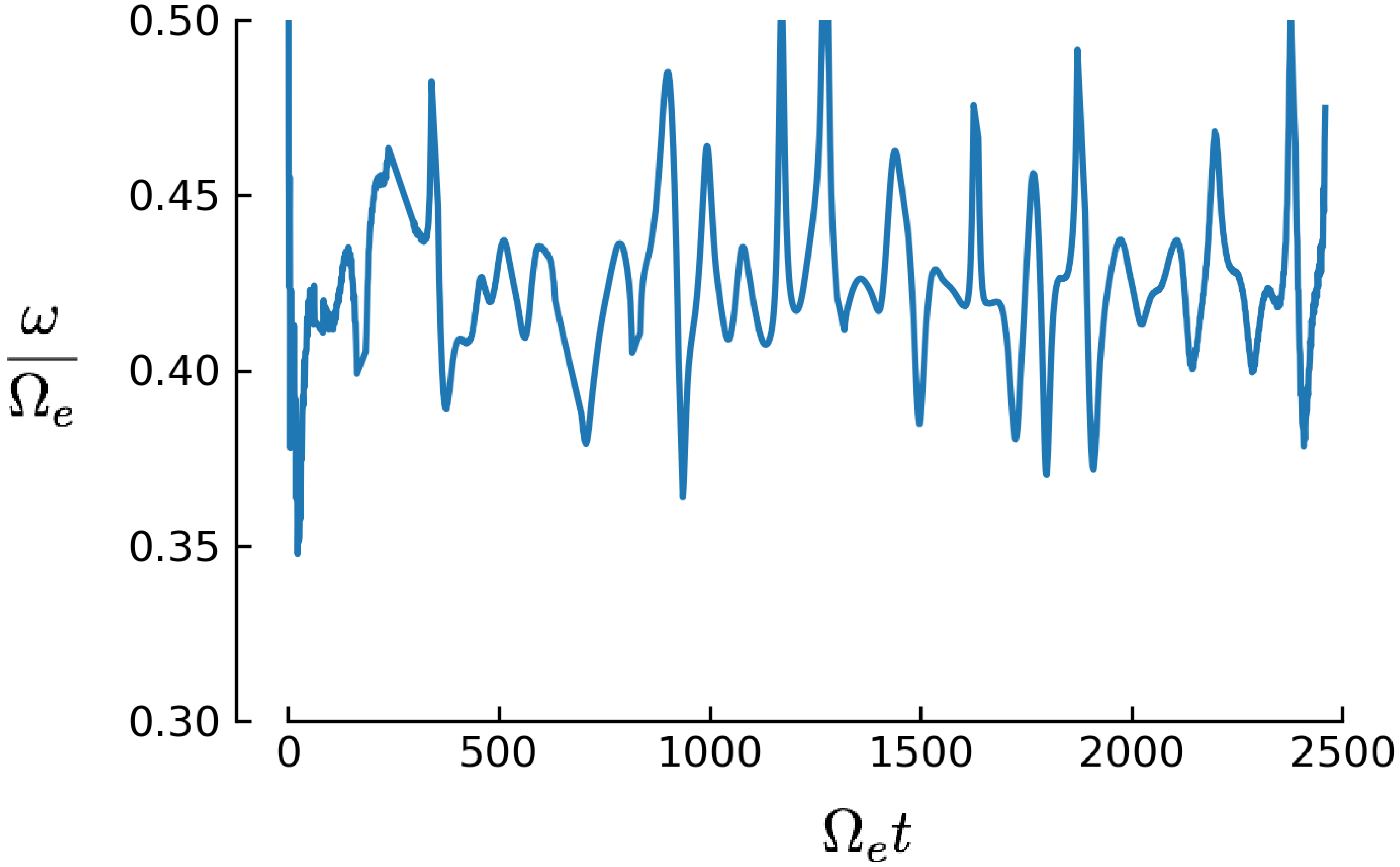


Electron beam experiments show triggered emissions

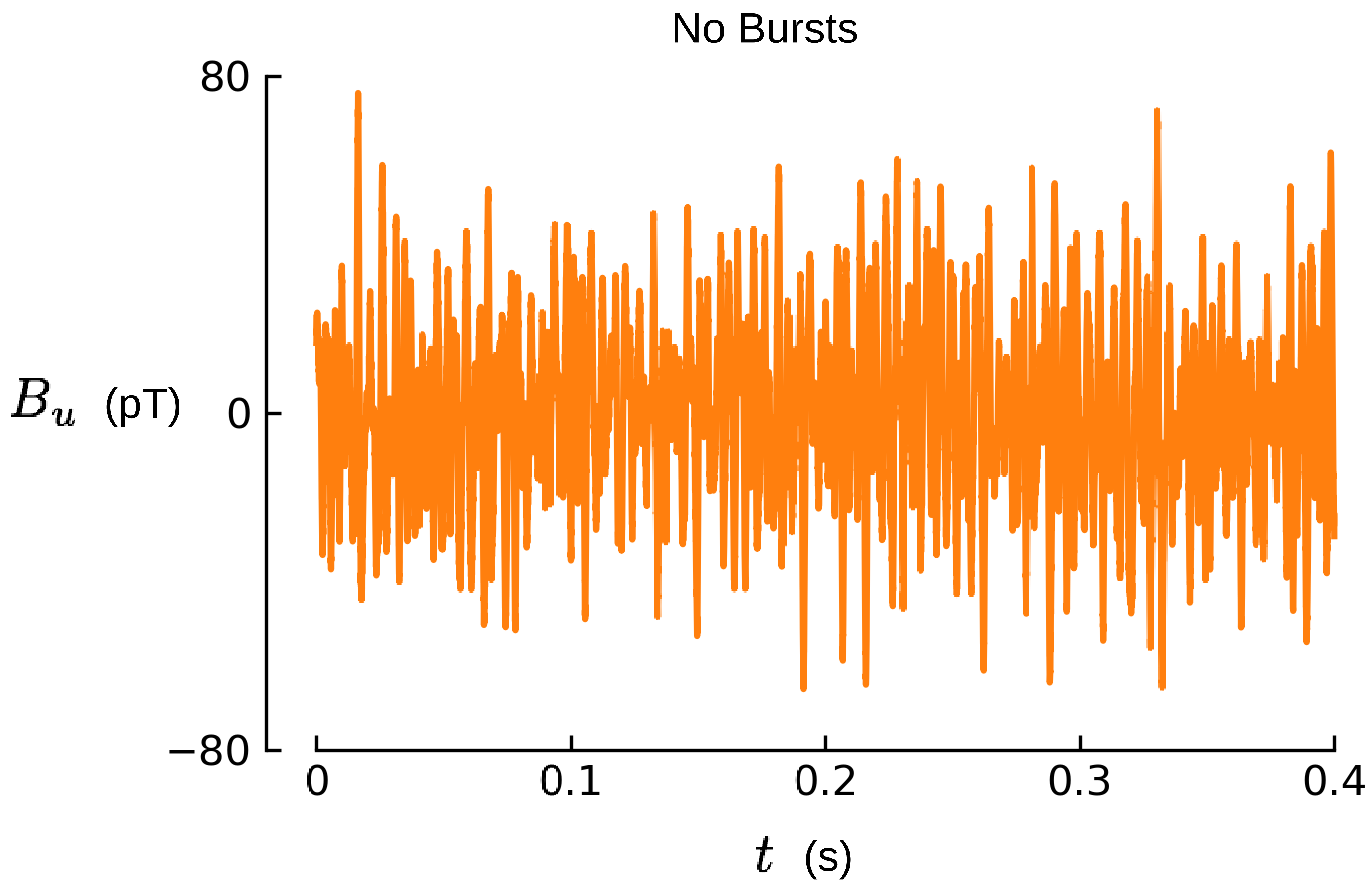


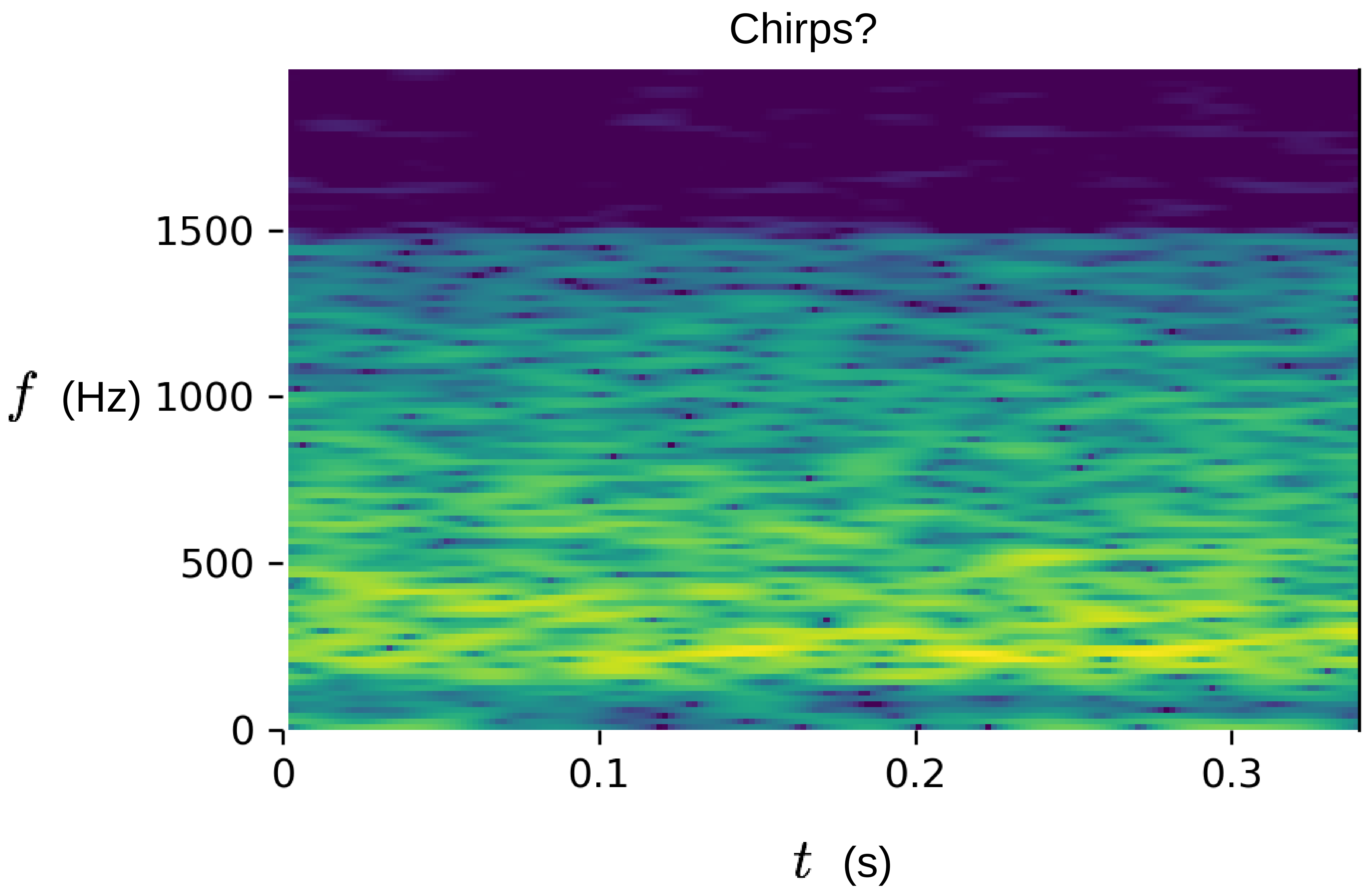
Electron beam driven whistlers show amplitude and phase modulations





Hiss does not have characteristics of chorus or electron beam experiments





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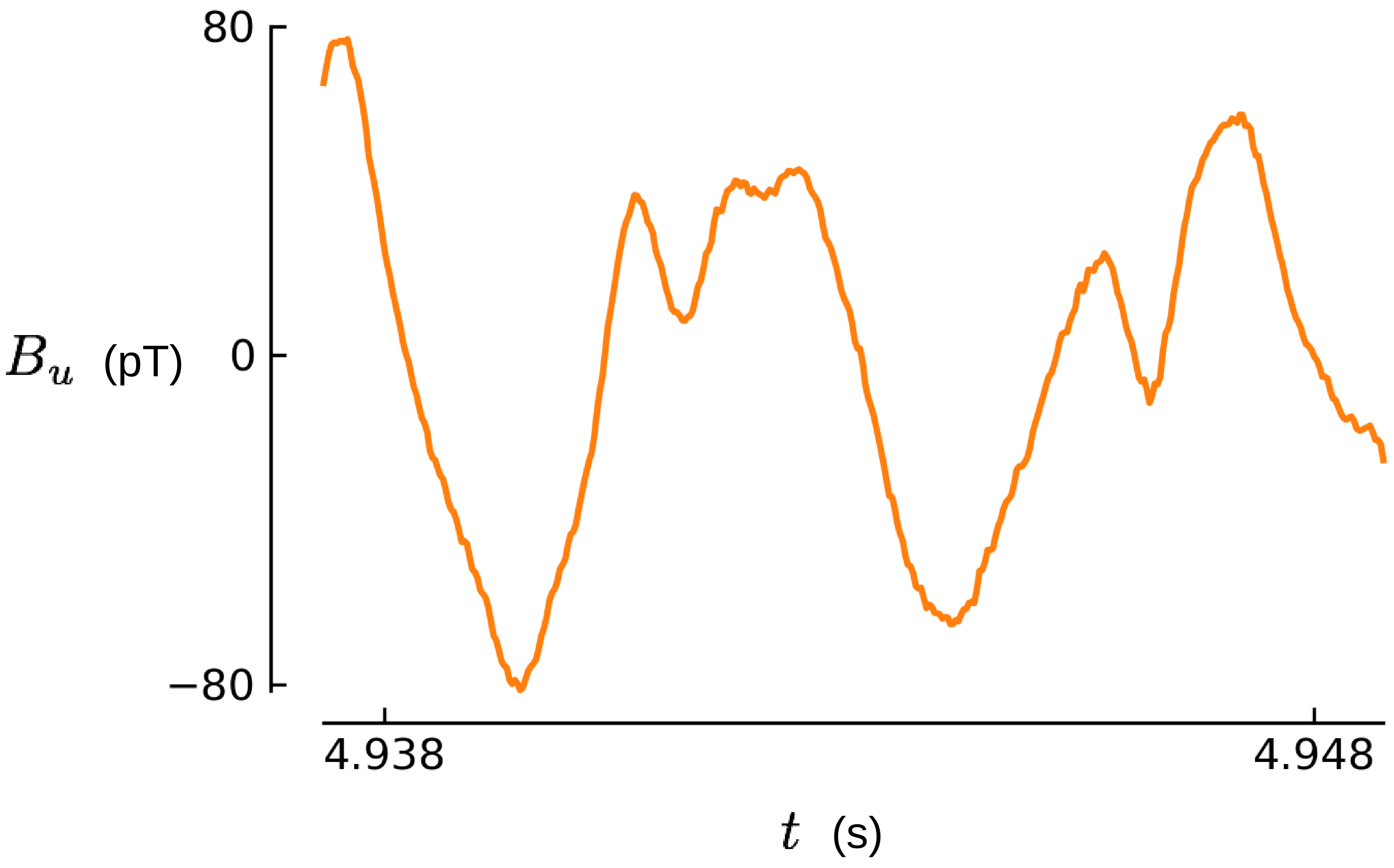
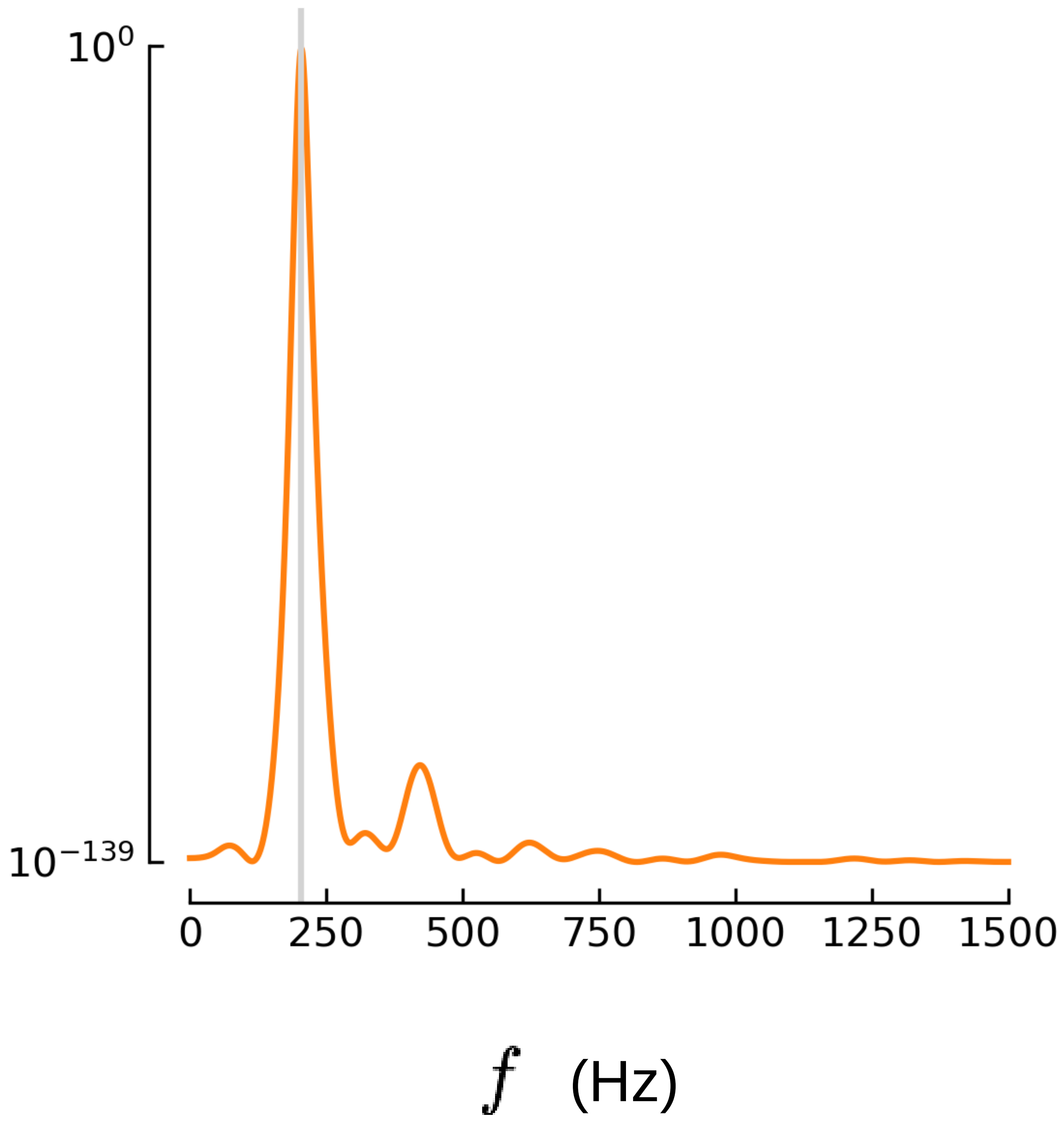
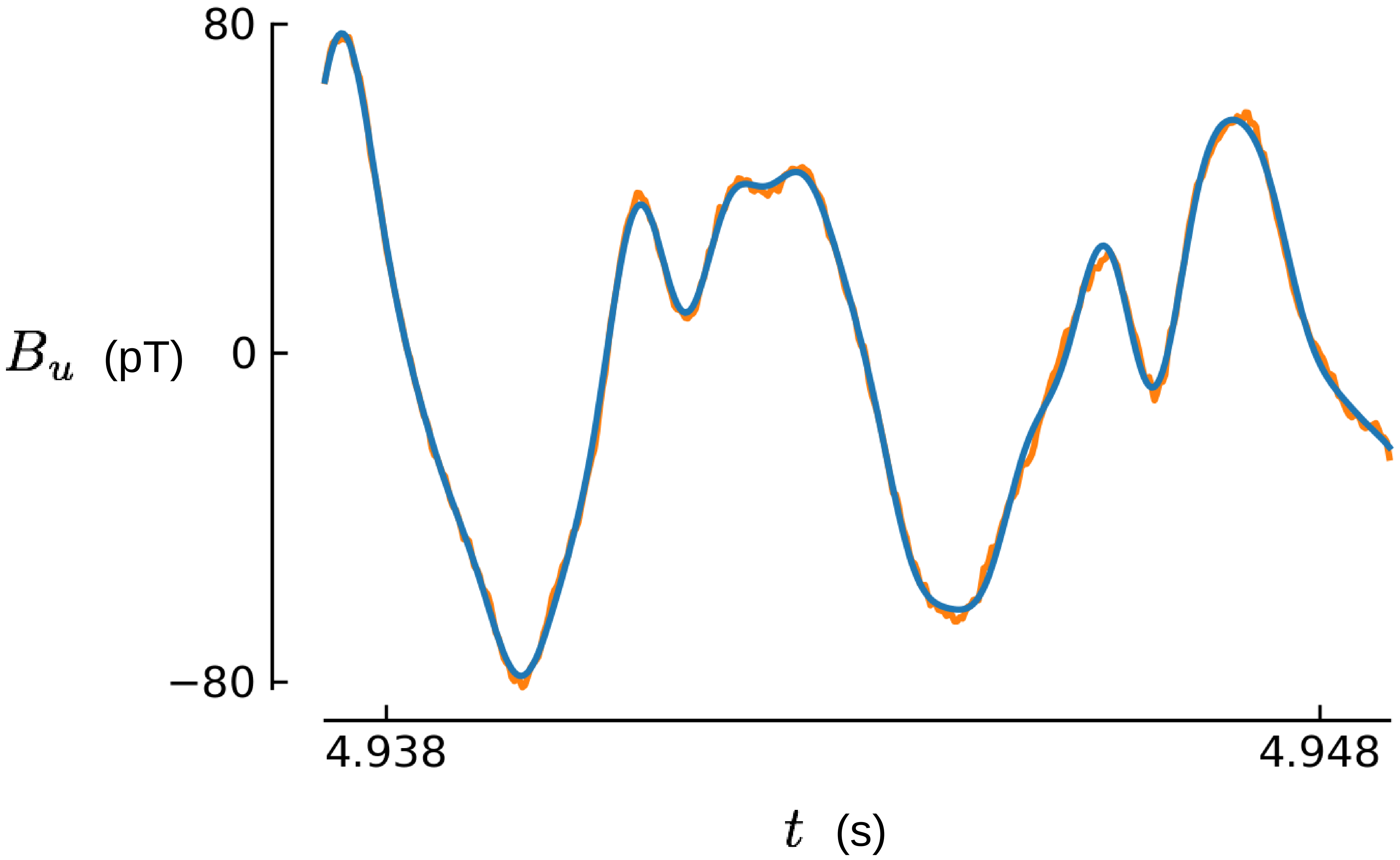
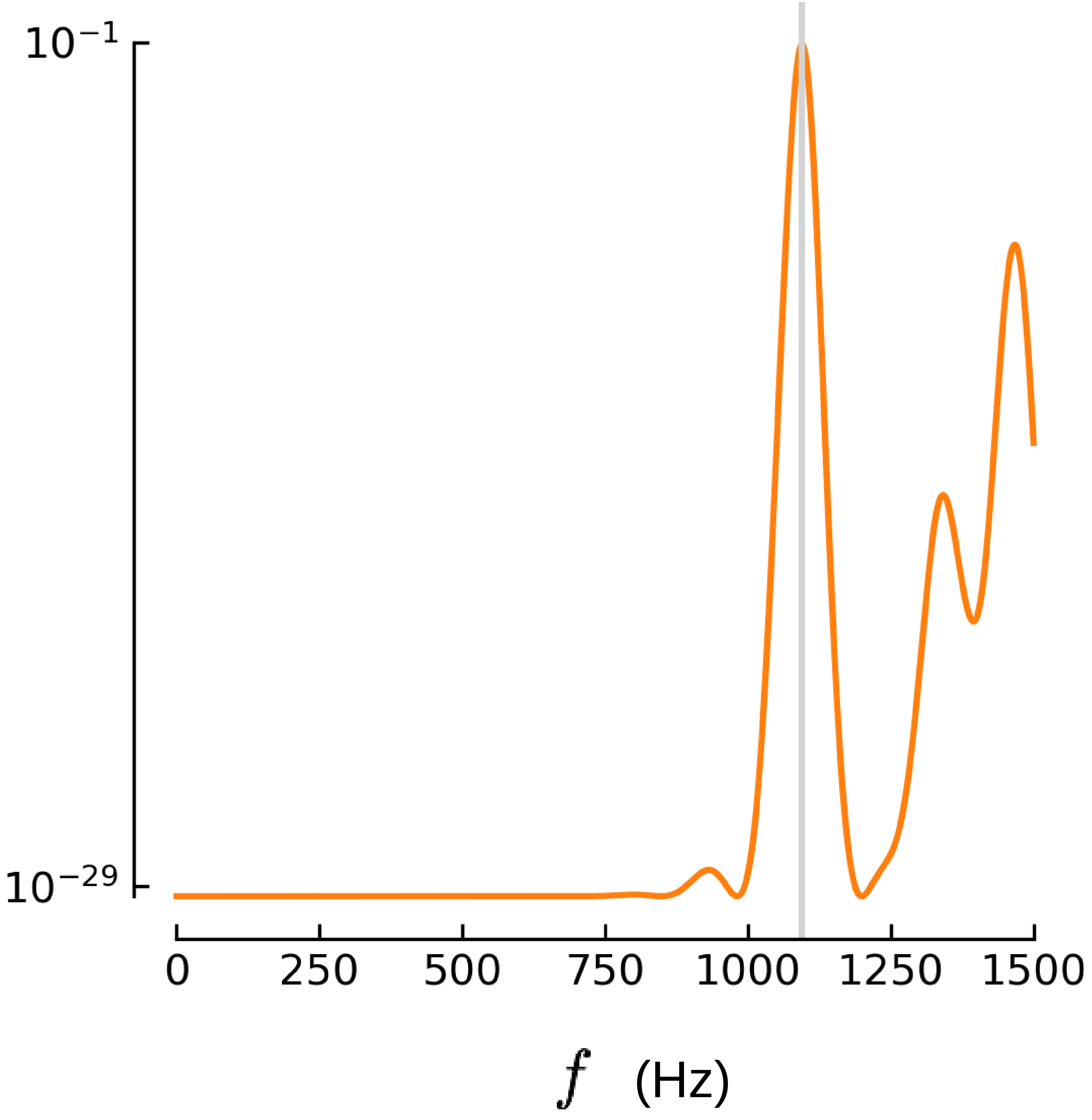




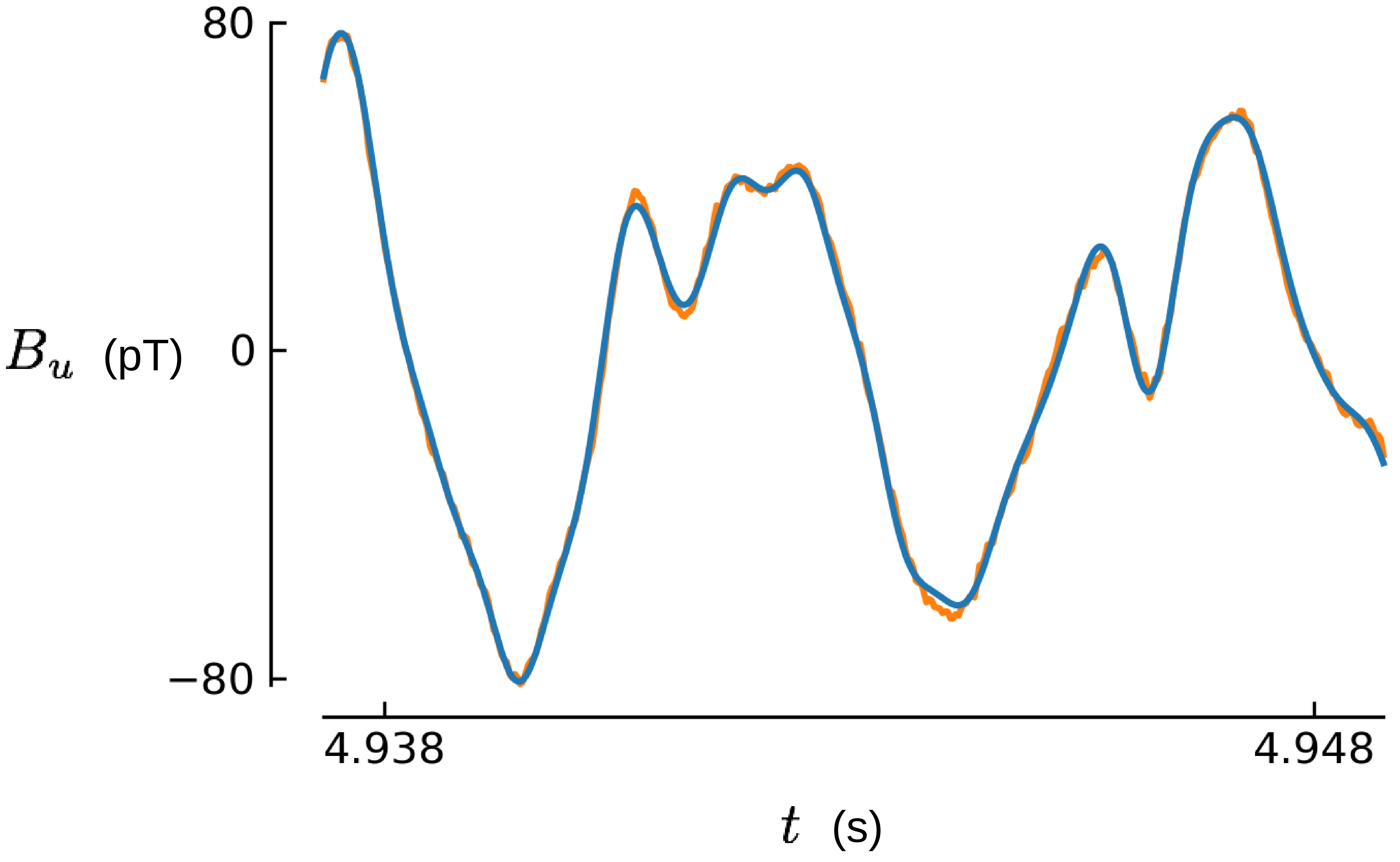
Hiss is probably due to sum of Fourier modes



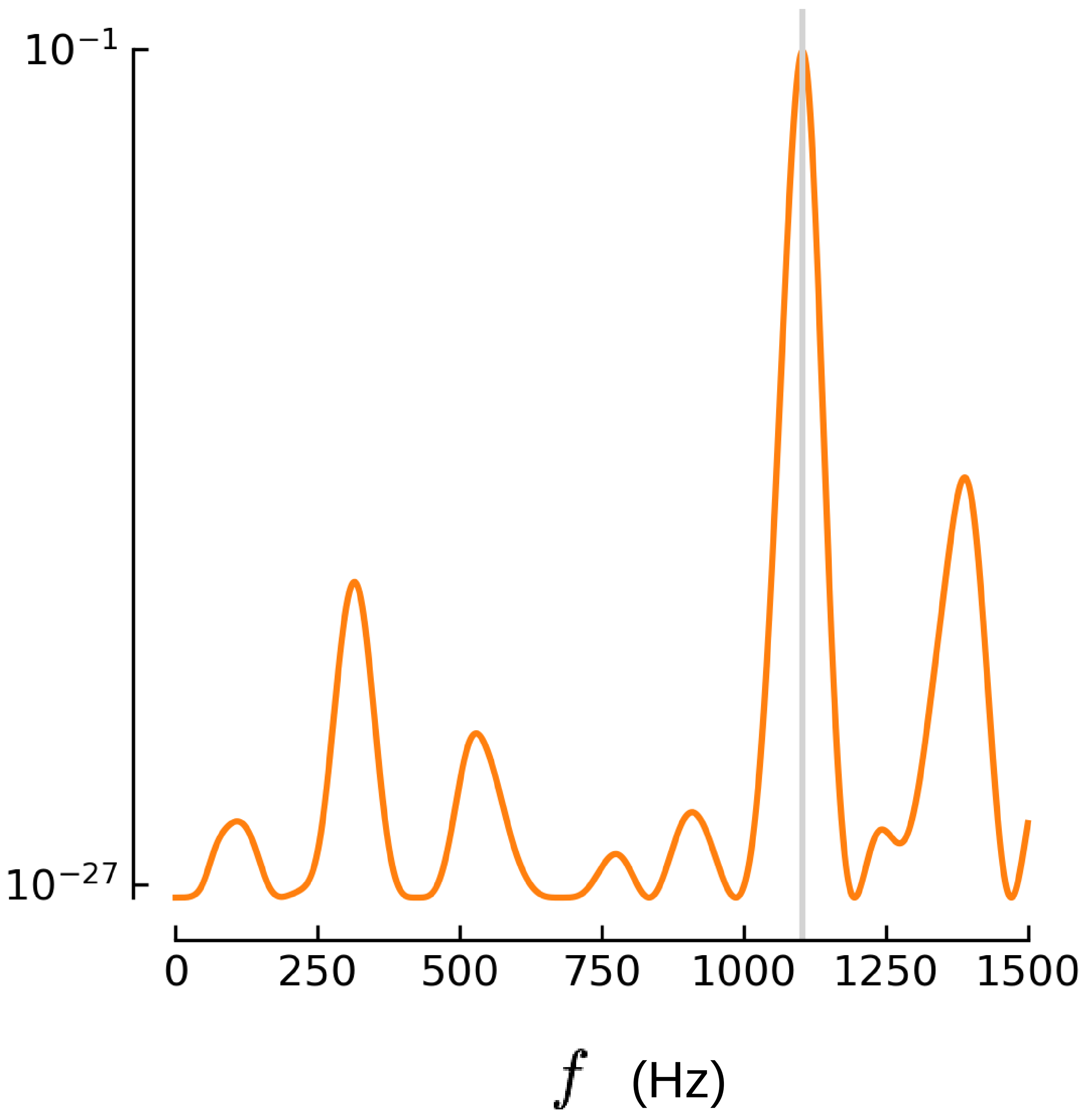


$$f_1(t) = \sum_{n=1}^{10} A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

Hiss is probably due to sum of Fourier modes



$$f_2(t) = A_0 e^{\gamma t}\cos(\omega t + \alpha t^2) + B_0 e^{\gamma t}\sin(\omega t + \alpha t^2) + \sum_{n=1}^{8} A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$



Hiss is probably due to sum of Fourier modes

Model	Relative Probability
$f_1(t) = \sum_{n=1}^{10} A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$	$P(f_1 \vert D) = 0.6$
$f_2(t) = A_0 e^{\gamma t} \cos(\omega t + \alpha t^2) + B_0 e^{\gamma t} \sin(\omega t + \alpha t^2) + \sum_{n=1}^{8} A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$	$P(f_2 \vert D) = 0.4$

Wave-form data can improve our understanding of nonlinear wave-particle **interactions**

Resonant particles control waveforms

- Conservation of momentum controls amplitude
- Conservation of energy controls frequency

Laboratory experiments

- Can test theories
- and give confidence to analysis techniques

Bayesian time-series techniques

- Give systematic approach to wave-form data
- Reveal coherent wave-particle interactions in Chorus and Laboratory triggered emissions and not in Hiss

We are only at the beginning stages
of understanding how to use wave-form data

More detailed physical theories are needed

More laboratory experiments are needed

Van Allen Probe has 4 terabytes of
high resolution waveform data

The astonishing reality of things
Is my discovery every day.
Each thing is what it is,
And it’s hard to explain to someone how happy this makes me,
And how much this suffices me.

–Alberto Caeiro 7 November 1915
heteronym of Fernando Pessoa